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Derivation and Simulation of Photon Trajectory from the Schwarzchild Metric

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ABSTRACT

Inspired by the German physicist, Von Laue (1879–1960), who was the first person to publish a diagram showing the trajectory of light so strongly affected by gravity, we present a visualization of photons near a compact object at the 'gravitational cutoff' formally known today as a black hole. We will derive the equations of motion for a photon orbiting a black hole directly from the Schwarzchild metric. The solutions can be derived by hand to a certain point, but in our case, we opted for numerical code to solve differential equations through iterative analysis. We will not only provide diagram's of the photon orbits, but also provide an animation that displays the time dilation of a photon from an observers perspective along with source code to run it on your own computer.

Keywords: Black Hole, Schwarzchild Metric, Schwarzchild Radius, Impact Parameter, Photon Sphere, Photon Trajectory

1. INTRODUCTION

A black hole is one of the compact objects that still involves various unsolvable mysteries as well as other types of compact objects such as neutron stars and white dwarfs. One of the main reasons why a black hole involves various unsolved problems is that light gets absorbed due to the strong gravitational field created by a black hole. Our observational methods have been dependent mostly on radiation detection in which we catch the radiation (photons) through observational instruments such as telescopes, antennae, and satellites from the universe, although new methods such as gravitational lensing and gravitational wave detection have provided exciting results recently. But once the light gets absorbed by a black hole, we can no longer see the information within the black hole thus making it harder to study what is going on inside the black hole.

Although the observational sides of the study of a black hole may need some improvements and developments, the theoretical sides of a black hole, however, has been studied by various physicists and astronomers to great length and developed it well enough to predict what is going on inside or around a black hole. During the era when people relied on the Newtonian frame of gravity, they did not predict the existence of compact objects because it does not explain the concept of change in time scale. After the release of Einstein's Theory of General Relativity (GR), Karl Schwarzschild, a German physicist and astronomer, solved Einsteins Field Equation and found a solution, called Schwarzschild Metric. That metric predicted the existence of a tiny compressed object with an escape velocity greater than the speed of light, meaning that these objects can not only just absorb light within a certain radius, the so-called Schwarzschild radius, but also make it impossible for any light to escaped the compact object. As a result of this mechanism, there are certain ranges from the center of that object to the Schwarzschild radius that appears to be black because light within the Schwarzschild radius does not reach the Earth to be detected; hence, it is called a black hole.

However, before Schwarzschild formally developed the concept of the black hole from the solution of GR and coined the term Schwarzschild radius, there was another German physicist, Max Von Laue, who predicted the light behavior around such a compact massive object. His interest lied in electromagnetic physics and discovered the diffraction of X-rays due to crystals for which

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he received the Nobel Physics Prize in 1914. Soon after Einstein published the GR, Von used the concept of GR and visualized the light (photon) paths around an object (which we call a black hole after Schwarzschild) in the second volume of his relativity textbook as seen in Figure 1.

Even though there was not an accurate and precise way to visualize the photon paths at the time of Von Laue, we have acquired the Schwarzschild Metric to formally visualize the photon paths around a black hole. With this as our motivation, we attempted to visualize the photon paths by solving the Schwarzschild Metric and using a programming platform, python.

2. ANALYSIS

Before we start to think of the paths of photons, it would be better to discuss the interaction between light and gravity. In the frame of Newtonian mechanics, the effect of gravity on light depends on how we define the model of light, even though the non-maximum limitation for the speed of light causes the path of light to be bent. With the introduction of General Relativity, the frame shifted to spacetime in which the coordinate is curved by massive objects and light traveling along this special coordinates. Therefore, as light or photons get closer to a massive object such as a star, then the light (photons) path would be bent by its gravitational effect. The accuracy of Einstein's theory of GR was confirmed by the observation of the bending of a light path around the Sun. Since the curved spacetime coordinates around a massive object determine the photons' paths, we can predict the photon trajectories by solving the Schwarzschild Metric. The heavier the object is, the more the light path would be bent around it. Therefore, light paths around a black hole would be curved substantially, and we could predict that the path of light would loop around the black hole at a certain distance from its center and appears to be a light circle, called a photon sphere, just as Von predicted on his textbook of relativity.

2.1. *Schwarzschild Metric for Photons*

As we consider photons traveling around the black hole a black hole (with the shape of a sphere and no spin), the metric is no longer a Euclidean flat space, and we need to consider a particular spacetime metric, the Schwarzschild Metric, the solution to the Einstein's theory of General Relativity

$$ds^2 = -(1 - \frac{r_s}{r})c^2 dt^2 + \frac{1}{(1 - \frac{rs}{r})} dr^2 + r^2 d\Omega^2 \qquad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2$, c is the speed of light ($c = 2.998 \times 10^8$ m/s), and r_s is the Schwarzschild radius of a black hole given by

$$r_s = \frac{2GM}{c^2} \qquad (2)$$

where M is a mass of a black hole and G is a gravitational constant. In general relativity, we define photons traveling at the speed of light as a light-like. Therefore, $ds = 0$, and proper time is zero. For simplicity, we'll consider photons that orbit around a black hole at an equatorial plane with fixed $\theta = \pi/2$, so $d\theta = 0$. As a result, the Schwarzschild Metric would be

$$0 = -(1 - \frac{r_s}{r})c^2 dt^2 + \frac{1}{(1 - \frac{r_s}{r})} dr^2 + r^2 d\phi^2 \qquad (3)$$

Although the proper time is zero, we could still express the constants of motions of specific energy (e) and specific angular momentum (ℓ) by introducing σ.

$$e = g_t \frac{dt}{d\sigma} \qquad (4)$$

$$\ell = g_\phi \frac{d\phi}{d\sigma} \qquad (5)$$

From 3, g_t is the coefficient of $-dt^2$ if g_t is independent of t and g_ϕ is the coefficient of $d\phi^2$ if g_ϕ is independent of ϕ. Plugging in these constants of motions and making a few arrangements of the Schwarzschild Metric, we derived the equation:

$$\frac{e^2}{c^2} = \frac{dr^2}{d\sigma^2} + \frac{\ell^2}{r^2} - \frac{\ell^2 r_s}{r^3} \qquad (6)$$

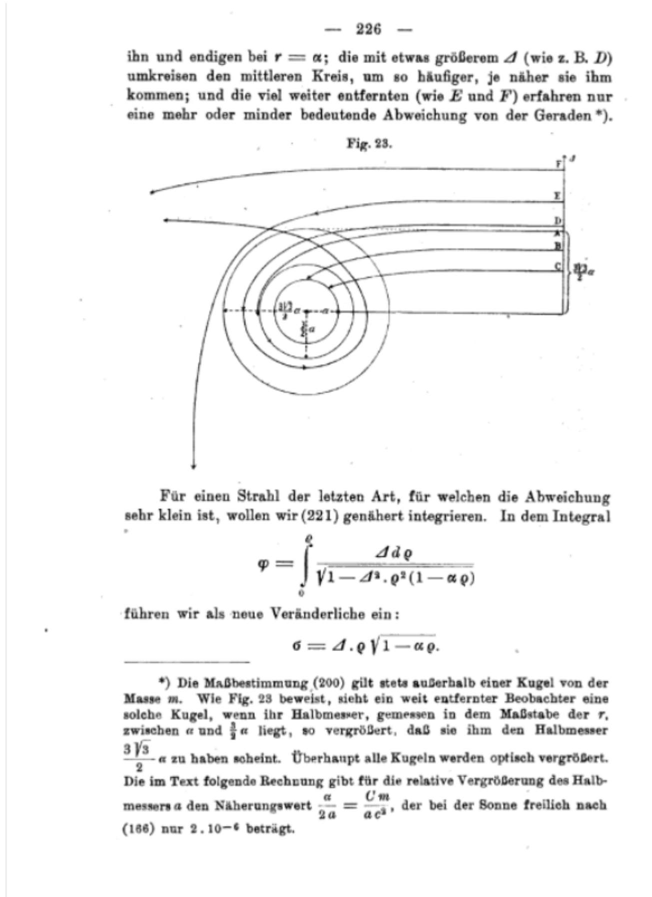
When r (the distance between a location of the object and a black hole) is fixed, $dr/d\sigma = 0$. We can solve for the ratio of the constant of motion e/ℓ, which by definition is another constant. That leads to the definition of impact parameter b

$$\frac{e}{\ell} = \frac{c}{b} \text{ where } b = r_m(1 - \frac{r_s}{r_m})^{-1/2} \qquad (7)$$

Also, from (6), we could define the effective potential as:

$$V_{\text{eff}}(r) = \frac{l^2}{r^2} - \frac{l^2 r_s}{r^3} \qquad (8)$$

By taking the first derivative of the effective potential and letting it zero, we could find where the extreme point of the orbit is, which is a photon sphere, $r = 3/2 r_s$. Also, taking the second derivative of it leads to a negative value, indicating the photon sphere is in an unstable orbit. Therefore, a photon does not stay in the orbit forever and would be either deflected of to infinity or fall into a black hole. Our objective for the final project is to visualize the photon trajectories around the black



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ihn und endigen bei $r = \alpha$; die mit etwas größerem Δ (wie z. B. D) umkreisen den mittleren Kreis, um so häufiger, je näher sie ihm kommen; und die viel weiter entfernten (wie E und F) erfahren nur eine mehr oder minder bedeutende Abweichung von der Geraden *).

Fig. 23.

Für einen Strahl der letzten Art, für welchen die Abweichung sehr klein ist, wollen wir (221) genähert integrieren. In dem Integral

$$\varphi = \int_0^\varrho \frac{\Delta \, d\varrho}{\sqrt{1 - \Delta^2 . \varrho^2 (1 - \alpha \varrho)}}$$

führen wir als neue Veränderliche ein:

$$\sigma = \Delta . \varrho \sqrt{1 - \alpha \varrho}.$$

*) Die Maßbestimmung (200) gilt stets außerhalb einer Kugel von der Masse m. Wie Fig. 23 beweist, sieht ein weit entfernter Beobachter eine solche Kugel, wenn ihr Halbmesser, gemessen in dem Maßstabe der r, zwischen α und $\frac{3}{2}\alpha$ liegt, so vergrößert, daß sie ihm den Halbmesser $\frac{3\sqrt{3}}{2}\alpha$ zu haben scheint. Überhaupt alle Kugeln werden optisch vergrößert. Die im Text folgende Rechnung gibt für die relative Vergrößerung des Halbmessers a den Näherungswert $\frac{\alpha}{2a} = \frac{Cm}{ac^2}$, der bei der Sonne freilich nach (166) nur $2 . 10^{-6}$ beträgt.

Figure 1. Von Laue This was the earliest diagram published by Von Laue in 1920 that depicts photons orbiting a compact object at the gravitational limit. Notice that at $r = 3/2 r_s$, there exists a photon sphere where photons can momentarily orbit the gravitationally compact object. Being a publication from the early 20th century, this figure was drawn semi-qualitatively, however, Von Laue understood the phenomenon well enough to predict that a photon's trajectory could not only be bent, but if incident at just the right impact parameter, could theoretically loop around the object. References: Von Laue 1923

hole. To do so, we could define the function of phi by taking the chain rule. By arranging the metric using this chain rule, we could express the function of phi as a differential equation

$$\frac{dr^2}{d\sigma^2} = \frac{dr^2}{d\phi^2}\frac{d\phi^2}{d\sigma^2} \tag{9}$$

Substituting 5 for $d\phi/d\sigma$, the result from 7 along with 9 into 6, we get the final result

$$\frac{dr}{d\phi} = r\sqrt{\frac{r^2}{b^2} - (1 - \frac{r_s}{r})} \tag{10}$$

While we can't solve this differential equation through trivial methods, we can implement iterative analysis as our method to solve for the photon trajectories. To do this, we set a small steps and some initial conditions to iterate, or otherwise, integrate through the entire photon trajectory.

$$r = r_0 + \frac{dr}{d\phi}d\phi \tag{11}$$

$$\phi = \phi_0 + d\phi \tag{12}$$

where dϕ is some small step with dϕ << 1 , and initial conditions to be chosen are r_0, ϕ_0, b, and r_s.

2.2. *Solving for Radial Direction as Photon Orbits*

There are four parameters for the initial conditions. To set the initial conditions accurately and appropriately, we first determined the parameter for the Schwarzschild radius because that is determined by the mass of a black hole we pick. When we look at the definition of the impact parameter b in 7, as we determine the value for the impact parameter b, r_0 would be automatically determined by solving it. To do so, we arranged the form of the definition of the impact parameter in the polynomial with variable r_m.

$$r_m^3 - b^2 r_m + b^2 r_s = 0 \tag{13}$$

This is a third degree polynomial. Typically, the cubic equation is shown as

$$ax^3 + bx^2 + cx + d = 0 \tag{14}$$

where x is a variable, and the other terms are constant. If we have a quadratic function, applying the quadratic formula can solve the two solutions. However, in order to solve a third-degree polynomial and find three solutions, we need to apply another formula, called Cardano's formula.

$$x_1 = S + T - \frac{b}{3a} \tag{15}$$

$$x_2 = -\frac{S+T}{2} - \frac{b}{3a} + \frac{i\sqrt{3}}{2}(S-T) \tag{16}$$

$$x_3 = -\frac{S+T}{2} - \frac{b}{3a} - \frac{i\sqrt{3}}{2}(S-T) \tag{17}$$

where

$$S = (R + \sqrt{Q^3 + R^2})^{\frac{1}{3}} \tag{18}$$

$$T = (R - \sqrt{Q^3 + R^2})^{\frac{1}{3}} \tag{19}$$

where

$$Q = \frac{3ac - b^2}{9a^2} \tag{20}$$

$$R = \frac{9abc - 27a^2d - 2b^3}{54a^3} \tag{21}$$

Since our cubic function (eq. (13)) derived from the definition of the impact parameter (eq.(6)) does not have a second degree, we can simplify Cardano's formula in terms of coefficients of eq. (13)

$$r_1 = S' + T' \tag{22}$$

$$r_2 = -\frac{S'+T'}{2} + \frac{i\sqrt{3}}{2}(S'-T') \tag{23}$$

$$r_3 = -\frac{S'+T'}{2} - \frac{i\sqrt{3}}{2}(S'-T') \tag{24}$$

where

$$S' = \left(-\frac{b^2r_s}{2} + \sqrt{\frac{(b^2r_s)^2}{4} + \frac{(-b^2)^3}{27}}\right)^{\frac{1}{3}} \tag{25}$$

$$T' = \left(-\frac{b^2r_s}{2} - \sqrt{\frac{(b^2r_s)^2}{4} + \frac{(-b^2)^3}{27}}\right)^{\frac{1}{3}} \tag{26}$$

The main solution to the radius for the initial condition should be from eq.(22) because eq.(23) and eq.(24) are complex solutions. Although all the solutions give the real part and imaginary part, the real part should be chosen as an initial condition for a radius.

2.3. *Visualization of Photon Trajectory by Python*

To visualize the photon sphere, we used the python program on the Jupyter Notebook. On the platform, we defined the necessary functions mentioned above. The next step was to determine the initial conditions. As mentioned earlier, we had to determine the value for the Schwarzschild radius first as it depends on the mass of a black hole we look at. After deciding the initial condition for the Schwarzschild radius, we can determine the values for a radius, which is the separation between the radial axis-aligned by a black hole and a parallel axis on which a photon moves, by various values for the impact parameters. These initial values calculated are shown in Table 1. For this project, we took the Schwarzschild radius of 1. Therefore, the photon sphere should be 1.5r$_s$. After substituting these initial conditions on the program that draws each photon's trajectory, the plot shows the photon path with an initial condition of 1.5r loops along the photon sphere. Also, we can observe that the path gets separated into two paths – either a deflected path or a path that falls into a black hole. This is a consequence of the property of the photon sphere's instability.

Initial Conditions

Radius	**Angle**	**Sch. Radius**	**Impact Parameter**
r$_0$	ϕ	r$_s$	b
1.01	0.967π	1	0.1
1.01	0.838π	1	0.5
1.01	0.667π	1	1
1.01	0.470π	1	1.5
1.01	0.197π	1	2
1.01	1.232π	1	2.598
1.5	0.902π	1	2.598
1.5	-0.167π	1	2.598
2.2267	$\pm$0.228π	1	3
2.8087	$\pm$0.325π	1	3.5
3.351	$\pm$0.375π	1	4
3.877	$\pm$0.400π	1	4.5
4.395	$\pm$0.417π	1	5
4.908	$\pm$0.428π	1	5.5
5.419	$\pm$0.443π	1	6
5.927	$\pm$0.455π	1	6.5
6.433	$\pm$0.460π	1	7

Table 1. This table shows the initial conditions for a radius, angle, Schwarzschild Radius, and impact parameter. The values for each radius were derived from Cardano's Formula by substituting the Schwarzschild Radius and impact parameter.

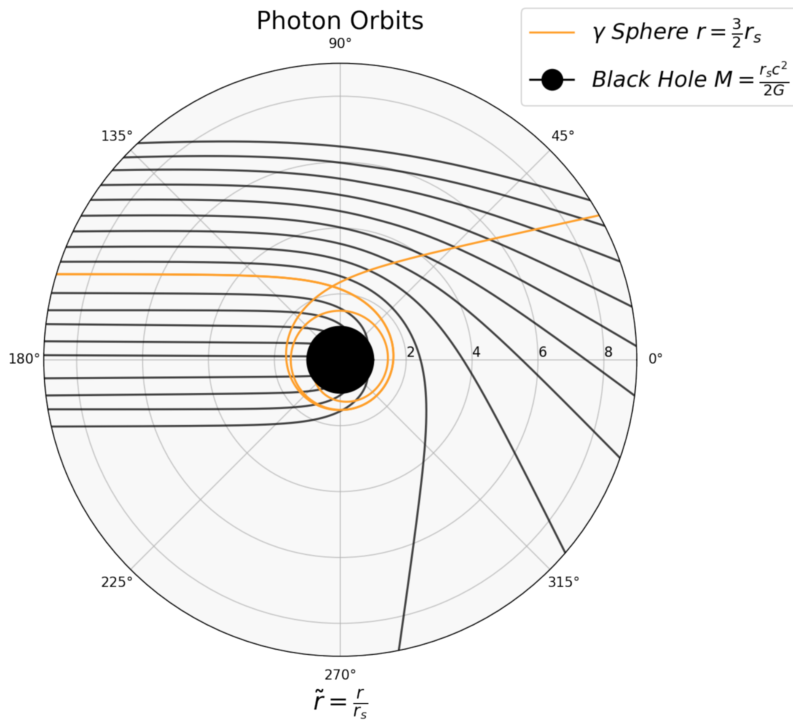

Figure 2. The Photons Orbits around a black hole.

This is a slice of the $\theta = \pi/2$ equitorial plane. We see the for a small enough impact parameter b, photons will simply fall into the black hole. Similarly, for large enough b, we find that photons are deflected by some angle ϕ. But the most interesting case, with $b \approx 2.5$ we find that photons can actually orbit around the black hole. This phenomenon is momentary and occurs at the photon sphere located at $r = 3/2r_S$ which was theorized by equation 8. Something to note is that this unstable orbit produces two cases. One where the photon simply falls in, and the other where the photon actually loops around the black hole and off infinity. These two trajectories are overlayed to highlight the split that can occur. The inital condition chosen for these trajectories can be found in Table 2.3

2.4. *The way forward*

While we have presented a fascinating visualization of photons orbiting a black hole, our job is not complete. This animation was only a slice on the $\theta = \pi/2$ equatorial plane of a black hole. Our original idea was to create a 3D projection onto a spherical plane have the photons come from various θ's. This could then be animating by similar methods to visualize the effect of gravitation lensing! The simplest case would be to create an Einstein ring where photons are emitted directly behind a lens such a black hole! However, upon starting the project equation 10 could not be integrated by trivial methods. After solivng for the equations of motion, our project turned into fine tuning initial conditions and animation. This project could go well beyond the scope of c161. For instance, we could use our photon trajectories to curve fit against a non-linear functions to write explicit equations for $r(\phi)$ for various b and r_s. Or this could go from a theoretical project into something in industry. The blockbuster hit movie, 'Interstellar' filmed by Christopher Nolan, called Astrophysicist's and computer scientist's to visualize a black hole. Their process was similar in that they numerically solved for $r(\phi)$ to create a spectacular visualisation that took over 100 hours to render. While our project was no where near as complex as that, the basic elements and approach presented here were also used to make a film.

3. CONCLUSIONS

Starting with the work done by Von as our motivation, we could visualize and animate the photon trajectories around a black hole and its loop around the photon sphere with a certain initial condition. The Schwarzschild Metric is successful in a way that we could interpret how the photon behaves outside the event horizon due to the General Relativity Effects. Even though it is difficult to solve the integral of the function of photons' orbits eq10, the simulation enables us to plot these orbits by determining the initial conditions and steps. As a result, we could see the properties of the photons' paths around a black hole in the simulation above. How-

Initial Conditions

Radius	**Angle**	**Sch. Radius**	**Impact Parameter**
r_0	ϕ	r_s	b
0.115	0.790π	1	0.1
0.190	0.735π	1	0.2
0.257	0.695π	1	0.3
0.380	0.637π	1	0.5
0.439	0.613π	1	0.6
0.496	0.590π	1	0.7
0.522	0.553π	1	0.8
0.608	0.548π	1	0.9
0.662	0.528π	1	1
0.930	0.427π	1	1.5
1.191	0.302π	1	2
1.256	0.260π	1	2.125
1.321	0.210π	1	2.25
1.450	0.013π	1	2.5

Table 2. This table shows the initial conditions for a radius, angle, Schwarzschild Radius, and impact parameter for Figure 5. The each angle is in a manner of $0.25n\pi$, where n = 1,2,3.... The values for each radius were derived from Cardano's Formula by substituting the Schwarzschild Radius and impact parameter.

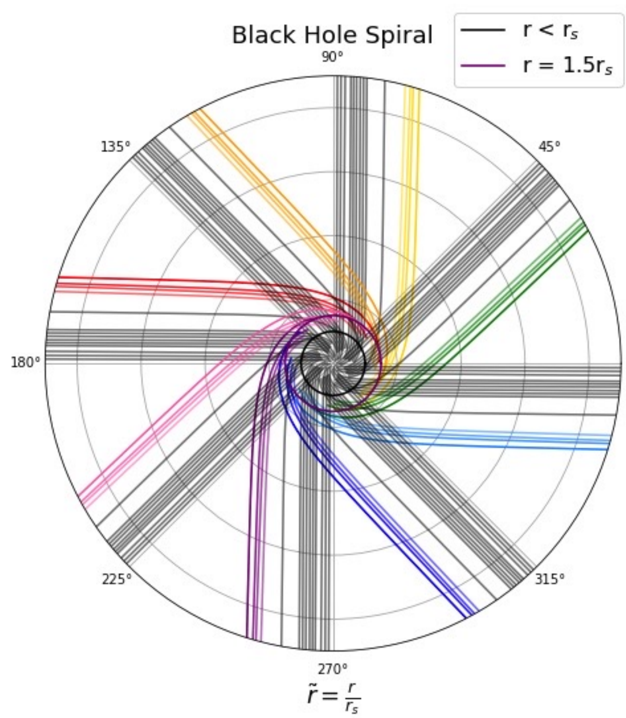

Figure 3. The Orbits of Photons on the range $r \leq 1.5\ r_s$ using the Cardano Formula

ever, the model on which this project focus is a special case in which a black hole has a spherical shape and does not spin itself. Thus solving the Schwarzschild Metric could deal with the processes above. If we start to use other types of black hole models, then we have to apply different metrics that have more parameters besides the Schwarzschild Metric. These different types of models of the black hole were not covered in this paper. Conceivable models of black holes are the following:

- One of the models is a black hole that spins itself. For simplicity, we chose the black hole without spin for this project. However, in reality, objects such as galaxies, stars, and planets tend to rotate as a natural property. That means the majority of black holes are considered to have a self-rotation. In this case, instead of applying the Schwarzschild Metric, the Kerr Metric, which is also found by Karl Schwarzschild, should be used.

- Another model of a black hole is an evaporating black hole. After the work by Stephen Hawking, he discovered that a black hole might emit radiation, called Hawking radiation. In this case, instead of the Schwarzschild Metric, another metric, called the Ingoing Vaidya Metric, would be used.

Facilities: KAIT Lounge (Cambell 525)

Software: Python Jupiter Notebook, Numpy, matplotlib, Math, Desmos

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